Some Eigenvalue Inequalities For a Class of Jacobi Matrices

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ABSTRACT

We consider the class of Jacobi (tridiagonal) matrices $\mathbf{T} = \mathbf{L} + \mathbf{D}$, where $\mathbf{L}$ is the negative of the discrete Laplacian and $\mathbf{D}$ is a diagonal matrix. We prove the inequality $\lambda_1(\mathbf{T}) \geqslant \lambda_1(\tilde{\mathbf{T}})$, where $\lambda_1(\mathbf{T})$ represents the lowest eigenvalue of the matrix $\mathbf{T}$ and where $\tilde{\mathbf{T}} = \mathbf{L} + \tilde{\mathbf{D}}$ with $\tilde{\mathbf{D}}$ being the "symmetric-increasing rearrangement" of $\mathbf{D}$. The proof follows from rearrangement inequalities going back at least to Hardy, Littlewood, and Pólya and is the one-dimensional discrete analogue of a well-known result for Schrödinger operators. We also prove that the gap, $\lambda_2 - \lambda_1$, is increased by strictly symmetric-increasing perturbations in the case that $\mathbf{D}$ is symmetric. Finally, we give an inequality relating the lowest eigenvalues of four Jacobi matrices of the form $\mathbf{T} = \mathbf{L} + \mathbf{D}$ when their potentials $\mathbf{D}$ obey certain conditions.

1. INTRODUCTION

In this paper we consider a number of problems concerning the eigenvalues of Jacobi (tridiagonal) matrices that are related to results concerning Schrödinger operators. Thus we shall focus attention on matrices of the form $\mathbf{T} = \mathbf{L} + \mathbf{D}$, where $\mathbf{L}$ is the negative of the discrete Laplacian and $\mathbf{D}$ is a diagonal matrix (the analogue of the potential in the case of Schrödinger

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operators). A number of these are direct analogues of recent results of ours [1–3].

In Sections 2 and 3 we show that $\lambda_1(\mathbf{T})$, the lowest eigenvalue of $\mathbf{T}$, is decreased (strictly speaking, not increased) by symmetric-increasing rearrangement of the matrix $\mathbf{D}$. This is the discrete analogue of a well-known result for Schrödinger operators (in any number of dimensions). The discrete version of this is a simple exercise based on general results contained in Hardy, Littlewood, and Pólya [4]. We choose to present our own proof of this inequality for two reasons. First, our proof is simple and direct and therefore somewhat more accessible than other proofs contained in the literature. Second, in giving our proof we take the opportunity of correcting certain minor misstatements about the case of equality in the result pertaining to the Laplacian alone which were made by Schwarz [5] and Lehman [6] in their discussions of this problem. In our proof the cases of equality are quite readily discerned.

In Section 4 we make some observations about how eigenvalue ratios and gaps behave when D is nonnegative and undergoes symmetric-increasing rearrangement. We analyze the low-order cases.

In Section 5 we restrict discussion to the case of "symmetric" $\mathbf{D}$'s, i.e. $\mathbf{D}$'s for which $d_{n+1-i} = d_i$ for $i = 1, 2, \ldots, n$, where n is the dimension of $\mathbf{D}$, and the d_i for $1 \leqslant i \leqslant n$ are the diagonal elements of $\mathbf{D}$. The main result of Section 5 is a sharp inequality for the gap between the two lowest eigenvalues of the matrix $\mathbf{T} = \mathbf{L} + \mathbf{D}$, where $\mathbf{D}$ is a symmetric, symmetric-increasing diagonal matrix.

Finally, in Section 6 we use techniques similar to those of Section 5 to prove a result comparing the lowest eigenvalues of four Jacobi matrices which are in a certain relation to each other. This result is useful in comparing how λ_1 changes under certain types of perturbations. In particular, the result yields a special case of the well-known result that λ_1 is a concave function along any line in the space of matrices $\mathbf{D}$. More generally, it says that λ_1 is strongly subadditive on the space of matrices $\mathbf{D}$ which are increasing (alternatively, decreasing). We shall discuss these notions further in Section 6.

2. REARRANGEMENTS AND THE DISCRETE LAPLACIAN.

We let $\mathbf{L}$ be the $n \times n$ discrete Laplacian: i.e. $\mathbf{L} \equiv [l_{ij}]$, $1 \leqslant i, j \leqslant n$, where

$$l_{ij} = \begin{cases} 2 & \text{if} \quad i = j, \\ -1 & \text{if} \quad |i-j| = 1, \\ 0 & \text{otherwise}. \end{cases} \tag{2.1}$$

With $x = (x_1, x_2, \dots, x_n) \in \Re^n$, the quadratic form $(x, \mathbf{L}x)$ is given by

$$(x, \mathbf{L}x) = x_1^2 + \sum_{i=1}^{n-1} (x_{i+1} - x_i)^2 + x_n^2. \tag{2.2}$$

Before stating the main result of this section we must define the notion of "symmetric-deceasing rearrangements" of a vector x.

DEFINITION. The *symmetric-decreasing rearrangement* [4] of $x \in \Re^n$ is the vector $x_R = (a_1, a_3, a_5, \dots, a_6, a_4, a_2) \in \Re^n$, where $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ is the listing of the x_i's in ascending order.

LEMMA 2.1. *Let $x \in \Re^n$ be a nonnegative vector (i.e. a vector with nonnegative entries), and let $\mathbf{L}$ be the discrete Laplacian (Equation 2.1). Then*

$$(x, \mathbf{L}x) \geqslant (x_R, \mathbf{L}x_R), \tag{2.3}$$

where x_R is the symmetric-decreasing rearrangement of x; ie., the kinetic-energy functional decreases under symmetric-decreasing rearrangement of its argument (for nonnegative vector arguments).

REMARK. In the continuum case a similar inequality holds. In that case, the kinetic-energy functional is replaced by the L_2-norm of the gradient of a positive function in H_1 (see e.g. [7]).

Proof. We argue that the quadratic form $(u, \mathbf{L}u)$ as given in Equation (2.2) takes its minimum at $u = x_R$ (not necessarily uniquely) when u is allowed to vary over all possible rearrangements of x. Obviously if $(u, \mathbf{L}u)$ is minimized at $u = x_R$, it will also be minimized at $u = B(x_R)$, where $B(x_R)$ denotes the vector obtained by reversing x_R [i.e. $B(x_R)_i = (x_R)_{n+1-i}$, $i = 1, 2, \dots, n$]. Let $i(x)$ be the least index i for which the entry a_i in x_R does not agree with the corresponding entry in x [if all entries agree, i.e. $x = x_R$, we set $i(x) = n+1$]. Similarly we define $i_B(x)$ by comparing terms of x and $B(x_R)$. By reversing x, if necessary, we may assume that $i(x) \geqslant i_B(x)$. If $i(x) = n+1$, then $x = x_R$ and we are done. If not, we shall show that by rearranging x we can increase $i(x)$ without increasing $(x, \mathbf{L}x)$. Since this argument can be repeated until $i(x) = n+1$, this will establish the lemma.

By the definition of $i = i(x)$, the entry in x which occupies the same position that a_i does in x_R is strictly larger than a_i. Let j be the position of the entry in the vector x having the value a_i (i.e. $x_j = a_i$). We now define the transformation $x \to \tilde{x}$ that increases i by at least 1 and that does not increase $(x, \mathbf{L}x)$. To define $\tilde{x}$ we consider four possibilities:

(1) i *odd*, $j \neq n-(i-1)/2$. Then define $\tilde{x} = [\tilde{x}_k]$ by

$$\tilde{x}_k = \begin{cases} x_j & \text{if } \; k = \dfrac{i+1}{2}, \\ x_{k-1} & \text{if } \; \dfrac{i+1}{2} < k \leqslant j, \\ x_k & \text{otherwise.} \end{cases}$$

Then

$$(\tilde{x}, \mathbf{L}\tilde{x}) - (x, \mathbf{L}x) = 2(x_{j+1} - x_j)(x_j - x_{j-1}) + 2(x_j - x_{(i-1)/2})(x_j - x_{(i+1)/2}) \leqslant 0,$$

since from the definition of i and x_j we have $x_{(i-1)/2} \leqslant x_j \leqslant x_{(i+1)/2}, x_{j-1}, x_{j+1}$.

(2) i *even*, $j \neq 1 + i/2$. Define $\tilde{x} = [\tilde{x}_k]$ by

$$\tilde{x}_k = \begin{cases} x_{k+1} & \text{if } \; j \leqslant k < n+1-\dfrac{i}{2}, \\ x_j & \text{if } \; k = n+1-\dfrac{i}{2}, \\ x_k & \text{otherwise.} \end{cases}$$

Then

$$(\tilde{x}, \mathbf{L}\tilde{x}) - (x, \mathbf{L}x) = 2(x_{j+1} - x_j)(x_j - x_{j-1}) + 2(x_j - x_{n+1-(i/2)})(x_j - x_{n+2-(i/2)}) \leqslant 0,$$

since from the definition of i and x_j we have

$$x_{n+2-(i/2)} \leqslant x_j \leqslant x_{n+1-(i/2)}, x_{j-1}, x_{j+1}.$$

(3) i *odd*, $j = n-(i-1)/2$. Then define $\tilde{x} = [\tilde{x}_k]$ by

$$\tilde{x}_k = \begin{cases} x_{n-k+1} & \text{if } \; \dfrac{i+1}{2} \leqslant k \leqslant j, \\ x_k & \text{otherwise.} \end{cases}$$

Then

$$(\tilde{x},\mathbf{L}\tilde{x})-(x,\mathbf{L}x)=2(x_{j+1}-x_{(i-1)/2})(x_j-x_{(i+1)/2})\leqslant 0,$$

since from the definition of i and x_j we have $x(i-1)/2 \leqslant x_{j+1} \leqslant x_j \leqslant x_{(i+1)/2}$.

(2) *i even*, $j=1+i/2$. Define $\tilde{x}=[\tilde{x}_k]$ by

$$\tilde{x}_k=\begin{cases} x_{n-k+2} & \text{if } \; j \leqslant k \leqslant n+1-i/2, \\ x_k & \text{otherwise.} \end{cases}$$

Then

$$(\tilde{x},\mathbf{L}\tilde{x})-(x,\mathbf{L}x)=2(x_j-x_{n+1-(i/2)})(x_{j-1}-x_{n+2-(i/2)})\leqslant 0,$$

since from the definition of i and x_j we have $x_{n+2-(i/2)} \leqslant x_{j-1} \leqslant x_j \leqslant x_{n+1-(i/2)}$. This proves the lemma. ∎

We remark that from our proof we can determine precisely which rearrangements of x minimize the quadratic form $(u,\mathbf{L}u)$. First, x_R and $B(x_R)$ always minimize $(u,\mathbf{L}u)$, so unless $x_R=B(x_R)$, the minimizer cannot be unique. Second, it is easy to see that if all the entries of x are distinct, then x_R and $B(x_R)$ are the only minimizers [since the rearrangements given in the proof above lead to strict decrease of $(u,\mathbf{L}u)$ in this case]. Third, any time two or more a_i's are equal, one can do the backwards ordering of the elements in the middle region [as described in cases (3) and (4) in the proof of Lemma 2.1] to obtain a (possibly) new minimizer of $(u,\mathbf{L}u)$. This possibility was missed by Schwarz [5] and Lehman [6]. They both state (in our notation) that if x has positive entries and no three elements of x are equal, then $(u,\mathbf{L}u)$ attains its minimum over rearrangements of x if and only if $u=x_R$ or $u=B(x_R)$ [5, p. 15; 6, p. 27]. This is incorrect, as is clear from our analysis above.

Lemma 2.1 has a simple extension to arbitrary vectors $x\in\mathfrak{R}^n$. First, let $|x|$ denote the vector whose entries are the absolute values of those of x. Then it follows from Lemma 2.1 that

$$(x,\mathbf{L}x)\geqslant(|x|,\mathbf{L}|x|)\geqslant(|x|_R,\mathbf{L}|x|_R), \tag{2.4}$$

since under $x \to |x|$, terms in $(x, \mathbf{L}x)$ from (2.2) are decreased if $x_i x_{i+1} < 0$ and remain the same if $x_i x_{i+1} \geqslant 0$.

3. THE RESULT $\lambda_1(\tilde{\mathbf{T}}) \leqslant \lambda_1(\mathbf{T})$

We now consider the more general Jacobi matrix $\mathbf{T} = \mathbf{L} + \mathbf{D}$ where $\mathbf{D}$ is a diagonal matrix $\mathbf{D} = \mathbf{diag}(d_1, d_2, \dots, d_n)$. Following the definition of the symmetric decreasing rearrangement of a vector given in Section 2, we define the *symmetric-increasing rearrangement* of $x \in \mathfrak{R}^n$ to be the vector $x^R = (b_1, b_3, \dots, b_4, b_2) \in \mathfrak{R}^n$, where $b_1 \geqslant b_2 \geqslant b_3 \geqslant \cdots \geqslant b_n$ is the listing of the entries of x in descending order. The matrix $\tilde{\mathbf{D}} = \mathbf{diag}(d^R)$, where $d = (d_1, d_2, \dots, d_n)$, will be called the *symmetric-increasing rearrangement* of $\mathbf{D}$. Let $\lambda_1(\mathbf{T})$ denote the least eigenvalue of the symmetric matrix $\mathbf{T}$.

THEOREM 3.1. *If* $\mathbf{T} = \mathbf{L} + \mathbf{D}$, *where* $\mathbf{L}$ *is the negative of the discrete Laplacian* (2.1) *and* $\mathbf{D}$ *is a diagonal matrix, then*

$$\lambda_1(\mathbf{T}) \geqslant \lambda_1(\tilde{\mathbf{T}}), \tag{3.1}$$

where $\tilde{\mathbf{T}} = \mathbf{L} + \tilde{\mathbf{D}}$ *with* $\tilde{\mathbf{D}}$ *being the symmetric-increasing rearrangement of* $\mathbf{D}$.

Proof. Let x represent a normalized eigenvector of $\mathbf{T}$ corresponding to the eigenvalue $\lambda_1(\mathbf{T})$. Then

$$\begin{aligned} \lambda_1(\mathbf{T}) &= (x, \mathbf{T}x) = (x, \mathbf{L}x) + (x, \mathbf{D}x) \\ &\geqslant (|x|_R, \mathbf{L}|x|_R) + (x, \mathbf{D}x), \end{aligned}$$

where the last inequality follows from the last remark in Section 2. Since $\mathbf{D}$ is diagonal, $(x, \mathbf{D}x) = (|x|, \mathbf{D}|x|)$. Let u be the vector whose entries are the elements of the diagonal of $\mathbf{D}$, and v the vector whose entries are the squares of the entries of x. Then

$$(|x|, \mathbf{D}|x|) = (u, v) \geqslant (u^R, v_R) = (|x|_R, \tilde{\mathbf{D}}|x|_R),$$

where we have used the result $(u, v) \geqslant (u^R, v_R)$, which is valid for arbitrary vectors $u, v \in \mathfrak{R}^n$ (see the proof of Theorem 368 of Hardy, Littlewood, and Pólya [4, Section 10.2, pp. 261–262], and note that the assumption that u and v have nonnegative entries is not needed). Therefore,

$$\lambda_1(\mathbf{T}) \geqslant (|x|_R, \mathbf{L}|x|_R) + (|x|_R, \tilde{\mathbf{D}}|x|_R) \geqslant \lambda_1(\tilde{\mathbf{T}}),$$

by the Rayleigh quotient characterization of $\lambda_1(\tilde{\mathbf{T}})$ [note that $(|x|_R, |x|_R) = (x, x) = 1$]. ∎

We remark that Schwarz's paper [5] contains a very similar analysis for the discrete analogue of the vibrating string (as opposed to the one-dimensional Schrödinger equation). The eigenvalues that he considers are those of $\mathbf{L}$ with respect to the weighted inner product $(u, \mathbf{P}u)$, where $\mathbf{P}$ is a positive diagonal matrix. The diagonal entries p_i of $\mathbf{P}$ may be thought of as the masses of n equally spaced particles on a massless string with constant tension and fixed ends. The eigenvalues are precisely the roots of the characteristic polynomial, $\det(\lambda\mathbf{P} - \mathbf{L})$, but, as above, it is more fruitful to use the variational characterization.

It is also of some interest to know which rearrangements of $\mathbf{D}$ (or $\mathbf{P}$) yield the minimal first eigenvalue. The result is that λ_1 is minimized if and only if $\mathbf{D}$ is $\tilde{\mathbf{D}}$ or $B(\tilde{\mathbf{D}})$. For Schwarz's problem the analogous result would be that $\mathbf{P}$ must be $\mathbf{P}^-$ or ${}^-\mathbf{P}$ (in his notation). This is also true, though because of his oversight regarding all possible minimizing rearrangements for $(u, \mathbf{L}u)$ his argument is not complete. Since the full arguments in both cases are rather long and not particularly instructive, we shall not go into them here. Our approach is to use concavity properties to show that the eigenvector v_1 for $\lambda_1(\tilde{\mathbf{D}})$ is first strictly increasing and then strictly decreasing, and hence it can assume a given value at most twice. From there one must get into detailed considerations of how internal backwards orderings which leave $(u, \mathbf{L}u)$ minimal might affect $(u, \mathbf{T}u)$. The upshot is that, due essentially to the rigidity of the recurrence relation and the inequalities defining a symmetric rearrangement, an allowed internal backwards ordering cannot change anything. That is, any section of v_1 lying above two equal entries must itself be symmetric. This is not as surprising as it might appear, since if this happens, the whole eigenvector must be symmetric (by the recursion relation). Thus we have even obtained the stronger result that if v_1, the first eigenvector for $\tilde{\mathbf{T}}$, has any two entries equal, it must be symmetric, as must $\tilde{\mathbf{D}}$.

4. OBSERVATIONS AND CONJECTURES ABOUT EIGENVALUE RATIOS

We have shown above that $\lambda_1(\mathbf{T})$ decreases (more precisely, does not increase) under symmetric-increasing rearrangements of $\mathbf{D}$. We have further conjectures on the behavior of certain ratios between eigenvalues of $\mathbf{T} = \mathbf{L} + \mathbf{D}$ under symmetric-increasing rearrangements of $\mathbf{D}$. These conjectures are based on numerical evidence and exact computations for the low-order cases.

We begin by stating the conjectures, and then we present some examples which illustrate and support them. Let $\mathbf{T} = \mathbf{L} + \mathbf{D}$ with $\mathbf{L}$ given by (2.1) and $\mathbf{D} = \mathbf{diag}(d_1, d_2, \ldots, d_n)$. Here we restrict $\mathbf{D}$ to satisfy $\min_i(d_i) = 0$. Let $\tilde{\mathbf{D}}$ denote the symmetric-increasing rearrangement of $\mathbf{D}$, and $\tilde{\mathbf{T}} = \mathbf{L} + \tilde{\mathbf{D}}$.

Conjecture 4.1.

$$\frac{\lambda_2}{\lambda_1}(\mathbf{T}) \leqslant \frac{\lambda_2}{\lambda_1}(\tilde{\mathbf{T}}).$$

While it is not true that λ_3/λ_1 always undergoes a change of definite sign under symmetric-increasing rearrangements of $\mathbf{D}$, there is numerical evidence to believe the following:

Conjecture 4.2.

$$\frac{\lambda_2 + \lambda_3}{\lambda_1}(\mathbf{T}) \leqslant \frac{\lambda_2 + \lambda_3}{\lambda_1}(\tilde{\mathbf{T}}).$$

Remarks.

(i) Although λ_1 does not increase under symmetric-increasing rearrangements of $\mathbf{D}$ (Theorem 3.1 above), it is not true that λ_2, λ_3, etc. undergo a change of definite sign under symmetric-increasing rearrangements (see Example 2 below).

(ii) While we believe that λ_2/λ_1 increases under symmetric-increasing rearrangements of $\mathbf{D}$ (Conjecture 4.1 above), it is not true that the eigenvalue gap $\lambda_2 - \lambda_1$ undergoes a change of definite sign under such rearrangements (see Example 3 below).

(iii) The conjectures above relate to earlier conjectures of Payne, Pólya, and Weinberger [8, 9] for the Laplacian on compact domains of different shapes with Dirichlet boundary conditions on $\mathfrak{R}^n$.

(iv) Similar conjectures could be formulated for the largest eigenvalues of $\mathbf{T}$ (i.e. combinations of λ_n, λ_{n-1}, etc.) under symmetric-decreasing rearrangements of $\mathbf{D}$ (see Section 7 below).

EXAMPLE 1 (Matrices of order 3). The general $\mathbf{T}$-matrix of order 3 is given by $\mathbf{T} = \mathbf{L} + \mathbf{D}$, with $\mathbf{D} = \mathbf{diag}(d_1, d_2, d_3)$ with $d_1, d_2, d_3 \geqslant 0$. The eigenvalues for this matrix $\mathbf{T}$ are the roots of the characteristic polynomial $P(\lambda) = \det(\lambda\mathbf{I} - \mathbf{T})$. It is easy to compute that $\tilde{P}(\lambda) = \det(\lambda\mathbf{I} - \tilde{\mathbf{T}}) = P(\lambda) + (d_2 - d_2^R) \geqslant P(\lambda)$; thus $P(\lambda)$ is displaced upwards by a constant term under symmetric-increasing rearrangements of $\mathbf{D}$. From this it follows that $\lambda_1(\tilde{\mathbf{T}}) \leqslant \lambda_1(\mathbf{T})$, $\lambda_2(\tilde{\mathbf{T}}) \geqslant \lambda_2(\mathbf{T})$, $\lambda_3(\tilde{\mathbf{T}}) \leqslant \lambda_3(\mathbf{T})$, while $\lambda_1 + \lambda_2 + \lambda_3$ remains the same under rearrangements. Therefore, $\lambda_2/\lambda_1(\tilde{\mathbf{T}}) \geqslant \lambda_2/\lambda_1(\mathbf{T})$ and $(\lambda_2 + \lambda_3)/\lambda_1(\tilde{\mathbf{T}}) \geqslant (\lambda_2 + \lambda_3)/\lambda_1(\mathbf{T})$. λ_3/λ_1 can go up or down under symmetric-increasing rearrangements, depending on the values of the d_i's. Take, for instance, $\mathbf{D} = \mathbf{diag}(2,2,0)$, which produces $\lambda_3/\lambda_1 = 3.40405$, while $\tilde{\mathbf{D}} = \mathbf{diag}(2,0,2)$ yields $\tilde{\lambda}_3/\tilde{\lambda}_1 = 3.73205 > \lambda_3/\lambda_1$. On the other hand, $\mathbf{D} = \mathbf{diag}(8,8,0)$ produces $\lambda_3/\lambda_1 = 5.89683$, while the rearranged $\tilde{\mathbf{D}} = \mathbf{diag}(8,0,8)$ yields $\tilde{\lambda}_3/\tilde{\lambda}_1 = 5.82840 < \lambda_3/\lambda_1$. *Note*: All numerical figures in these examples are expressed with five decimal places.

EXAMPLE 2 (Behavior of higher eigenvalues under symmetric-increasing rearrangements). For $n \times n$ $\mathbf{T}$-matrices (with $n > 3$), no eigenvalue except the lowest undergoes a change of definite sign under symmetric-increasing rearrangements. Take for example $\mathbf{D} = \mathbf{diag}(0,0,8,8,0)$, which has $\lambda_2 = 1.87506$, while $\tilde{\mathbf{D}} = \mathbf{diag}(8,0,0,0,8)$ has $\tilde{\lambda}_2 = 1.87689 > \lambda_2$. On the other hand, $\mathbf{D} = \mathbf{diag}(8,8,0,0,0)$ yields $\lambda_2 = 1.93714 > \tilde{\lambda}_2$. Another example illustrating this fact is $\mathbf{D} = \mathbf{diag}(3,6,2,0,4)$, which has $\lambda_2 = 4.04657$,while $\tilde{\mathbf{D}} = \mathbf{diag}(6,3,0,2,4)$ has $\tilde{\lambda}_2 = 3.86613 < \lambda_2$. On the other hand, $\mathbf{D} = \mathbf{diag}(0,6,2,4,3)$ has $\lambda_2 = 3.33201 < \tilde{\lambda}_2$. One of the examples given above also illustrates the case for λ_3. Take $\mathbf{D} = \mathbf{diag}(0,0,8,8,0)$, which has $\lambda_3 = 2.93061$, while $\tilde{\mathbf{D}} = \mathbf{diag}(8,0,0,0,8)$ has $\tilde{\lambda}_3 = 3.34112 > \lambda_3$. On the other hand, $\mathbf{D} = \mathbf{diag}(8,8,0,0,0)$ has $\lambda_3 = 3.37810 > \tilde{\lambda}_3$.

EXAMPLE 3 (Behavior of $\lambda_2 - \lambda_1$ under symmetric-increasing rearrangements of $\mathbf{D}$). For 3×3 $\mathbf{T}$-matrices, $\lambda_2 - \lambda_1$ always goes up under symmetric-increasing rearrangements of $\mathbf{D}$, as we have shown in Example 1.

However, for $n \times n$ **T**-matrices $\lambda_2 - \lambda_1$ does not exhibit a change of definite sign under these rearrangements. Consider for example $\mathbf{D} = \mathbf{diag}(8,8,0,0,0)$, which has $\lambda_2 - \lambda_1 = 1.37944$, while $\tilde{\mathbf{D}} = \mathbf{diag}(8,0,0,0,8)$ has $\tilde{\lambda}_2 - \tilde{\lambda}_1 = 1.34490 < \lambda_2 - \lambda_1$. On the other hand $\mathbf{D} = \mathbf{diag}(0,0,8,8,0)$ has $\lambda_2 - \lambda_1 = 0.93259 < \tilde{\lambda}_2 - \tilde{\lambda}_1$.

EXAMPLE 4 (Some numerical evidence supporting Conjectures 4.1 and 4.2). We have computed the eigenvalue ratios λ_2/λ_1 and $(\lambda_2 + \lambda_3)/\lambda_1$ for a large number of **T**-matrices of different dimensions. In all cases we have explored, these ratios satisfy the statements of Conjectures 4.1 and 4.2. For 3×3 matrices, these conjectures are true, as we have shown in Example 1. Here we will only give three numerical examples to illustrate the contents of the conjectures. The examples we have chosen here are the most unfavorable ones, in the sense that $\lambda_2 - \lambda_1 > \tilde{\lambda}_2 - \tilde{\lambda}_1$ for all of them. Consider first $\mathbf{D} = \mathbf{diag}(8,8,0,0,0)$, which has $\lambda_2/\lambda_1 = 3.47343$, while $\tilde{\mathbf{D}} = \mathbf{diag}(8,0,0,0,8)$ has $\tilde{\lambda}_2/\tilde{\lambda}_1 = 3.52805 > \lambda_2/\lambda_1$. Also, for this example we have $(\lambda_2 + \lambda_3)/\lambda_1 = 9.53060 < 9.80845 = (\tilde{\lambda}_2 + \tilde{\lambda}_3)/\tilde{\lambda}_1$. As a second example, consider $\mathbf{D} = \mathbf{diag}(8,8,8,8,8,8,8,8,0,0)$, which has $\lambda_2/\lambda_1 = 3.10903$, while $\tilde{\mathbf{D}} = \mathbf{diag}(8,8,8,8,0,0,8,8,8,8)$ has $\tilde{\lambda}_2/\tilde{\lambda}_1 = 3.21429 > \lambda_2/\lambda_1$. Also, for this example we have $(\lambda_2 + \lambda_3)/\lambda_1 = 11.72873 < 12.66350 = (\tilde{\lambda}_2 + \tilde{\lambda}_3)/\tilde{\lambda}_1$. As a last example, consider $\mathbf{D} = \mathbf{diag}(0,10,0,0,10,0)$, which has $\lambda_2/\lambda_1 = 2.09725$, while $\tilde{\mathbf{D}} = \mathbf{diag}(10,0,0,0,0,10)$ has $\tilde{\lambda}_2/\tilde{\lambda}_1 = 3.67735 > \lambda_2/\lambda_1$. Also, for this example we have $(\lambda_2 + \lambda_3)/\lambda_1 = 4.21582 < 10.78445 = (\tilde{\lambda}_2 + \tilde{\lambda}_3)/\tilde{\lambda}_1$.

5. BOUNDS FOR EIGENVALUE GAPS

In this section we focus attention on the gap $\lambda_2 - \lambda_1$ between the two lowest eigenvalues of Jacobi matrices $\mathbf{T} = \mathbf{L} + \mathbf{D}$, where $\mathbf{D}$ is now required to be symmetric (i.e. $d_i = d_{n+1-i}$). We prove results analogous to our results in [1] (see also [3]) for the continuous case. The general approach here is very similar to the approach used in that case. Thus we begin by proving a general comparison theorem which allows us to compare the gaps of two matrices $\mathbf{T}_0 = \mathbf{L} + \mathbf{D}_0$ and $\mathbf{T}_1 = \mathbf{L} + \mathbf{D}_1$. We think of $\mathbf{T}_1$ as obtained from $\mathbf{T}_0$ through addition of the perturbation $\mathbf{D}_1 - \mathbf{D}_0$. We then have:

THEOREM 5.1. *Let $\mathbf{T}_i = \mathbf{L} + \mathbf{D}_i$ for $i = 0,1$ be Jacobi matrices as defined above, i.e. with each $\mathbf{D}_i$ a symmetric diagonal matrix. Let $\lambda_1(\mathbf{T}_i)$ and $\lambda_2(\mathbf{T}_i)$ denote the first and second eigenvalues of $\mathbf{T}_i$, $i = 0,1$. If $\mathbf{D}_2 - \mathbf{D}_1$ is* *symmetric-increasing, then*

$$\lambda_2(\mathbf{T}_1)-\lambda_1(\mathbf{T}_1)\geqslant\lambda_2(\mathbf{T}_0)-\lambda_1(\mathbf{T}_0). \tag{5.1}$$

Furthermore, equality holds if and only if $\mathbf{D}_1-\mathbf{D}_0$ *is a constant times the identity matrix.*

Using the fact that the eigenvalues of $\mathbf{L}$ are given by

$$\lambda_k(\mathbf{L})=4\sin^2\left(\frac{k\pi}{2(n+1)}\right)=2\left[1-\cos\left(\frac{k\pi}{n+1}\right)\right] \tag{5.2}$$

for $k=1,2,\ldots,n$, we obtain the following result by setting $\mathbf{D}_0=0$:

COROLLARY 5.2. *The first two eigenvalues of* $\mathbf{T}=\mathbf{L}+\mathbf{D}$ *obey*

$$\lambda_2-\lambda_1\geqslant 2\left[\cos\left(\frac{\pi}{n+1}\right)-\cos\left(\frac{2\pi}{n+1}\right)\right] \tag{5.3}$$

if $\mathbf{D}$ *is a symmetric, symmetric-increasing diagonal matrix. Equality holds in* (5.3) *if and only if* $\mathbf{D}$ *is a constant times the identity matrix.*

REMARK. These results show that symmetric, symmetric-increasing perturbations tend to increase the gap $\lambda_2-\lambda_1$.

Proof of Theorem 5.1. We begin by fixing notation. Let the first two eigenvalues and eigenvectors of $\mathbf{T}_0$ be denoted by μ_1,μ_2 and v_1,v_2, and let those of $\mathbf{T}_1$ be denoted λ_1,λ_2 and u_1,u_2. Thus

$$\mathbf{T}_0v_k=\mu_kv_k \qquad \text{for} \quad k=1,2, \tag{5.4}$$

and

$$\mathbf{T}_1u_k=\lambda_ku_k \qquad \text{for} \quad k=1,2. \tag{5.5}$$

The proof is based on using three of the eigenvectors to construct a suitable trial vector for the fourth to be used in estimating the fourth eigenvalue via the Rayleigh-Ritz inequality. To insure that the trial vector is a suitable one, we shall need the symmetry properties of the eigenvectors of the $\mathbf{T}$'s, which are a consequence of the symmetry of the $\mathbf{D}$'s. The symmetry of the $\mathbf{D}$'s implies that v_1 and u_1 are symmetric and v_2 and u_2 are antisymmetric with respect to reversal of indices (i.e. $i \to n+1-i$) [10, 11]. More formally, with the parity matrix $\mathbf{J}$ defined to have ones along its antidiagonal and zeros elsewhere, the result is that since $\mathbf{J}^{-1}\mathbf{T}_0\mathbf{J} = \mathbf{T}_0$ and $\mathbf{T}_0$ is self-adjoint, a complete set of eigenvectors v_k of $\mathbf{T}_0$ corresponding to the eigenvalues μ_k (where $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n$ is a listing, with multiplicities, of the eigenvalues of $\mathbf{T}_0$) may be chosen to obey $\mathbf{J}v_k = (-1)^{k+1}v_k$ for $1 \leqslant k \leqslant n$, and similarly for eigenvectors u_k of $\mathbf{T}_1$. Moreover, the eigenvectors v_k (and similarly, u_k) can be chosen to have exactly $k-1$ sign changes. This is a classical property of discrete Sturm-Liouville eigenvalue problems. One can prove it as an application of Sturm sequences (for counting the zeros of a polynomial) from the theory of equations (see [12] for the general theory of Sturm sequences, [14] for a discussion relating directly to tridiagonal matrices, and [15] for a complete but rather terse summary of the relevant facts concerning tridiagonal matrices). These conditions force the eigenvectors u_2 and v_2 to change sign only at their centers.

Take

$$v = \frac{u_2 v_1}{u_1} \tag{5.6}$$

as a trial function for v_2. In (5.6) the operations are to be carried out entrywise. This is a well-defined expression, since u_1 (and v_1) can be taken with all positive entries (which we do henceforth). The symmetry of the $\mathbf{D}$'s implies that v is orthogonal to v_1 (v_1 being symmetric and v antisymmetric). We can then make use of the Rayleigh-Ritz inequality in the form

$$\mu_2 \leqslant \frac{(v, \mathbf{T}_0 v)}{(v, v)} \qquad \text{for} \quad v \perp v_1 \text{ and } v \neq 0. \tag{5.7}$$

Continuing with the main part of the proof, we let d_i^k denote the diagonal elements of $\mathbf{D}_k$ for $k = 0, 1$ and compute (by convention, we take $x_0 = x_{n+1} = 0$ for any vector x considered)

$$\begin{aligned}
(\mathbf{T}_0 v)_i &= -v_{i-1} + (2 + d_i^0)v_i - v_{i+1} \\
&= -u_{2,i-1}\frac{v_{1,i-1}}{u_{1,i-1}} + (2 + d_i^0)\frac{u_{2,i}v_{1,i}}{u_{1,i}} - \frac{u_{2,i+1}v_{1,i+1}}{u_{1,i+1}} \\
&= \frac{u_{2,i}}{u_{1,i}}\left[-v_{1,i-1} + (2 + d_i^0)v_{1,i} - v_{1,i+1}\right] + \frac{u_{2,i}v_{1,i-1}}{u_{1,i}} \\
&\quad + \frac{u_{2,i}v_{1,i+1}}{u_{1,i}} - \frac{u_{2,i-1}v_{1,i-1}}{u_{1,i-1}} - \frac{u_{2,i+1}v_{1,i+1}}{u_{1,i+1}}.
\end{aligned}$$

After further computations, using the recursion relations for v_1, u_1, and u_2, one finds that

$$(\mathbf{T}_0 v)_i = (\mu_1 + \lambda_2 - \lambda_1)v_i - u_{1,i-1}\Delta\left(\frac{v_{1,i-1}}{u_{1,i-1}}\right)\Delta\left(\frac{u_{2,i-1}}{u_{1,i-1}}\right) - u_{1,i+1}\Delta\left(\frac{v_{1,i}}{u_{1,i}}\right)\Delta\left(\frac{u_{2,i}}{u_{1,i}}\right),$$

where the second term is absent when $i = 1$ and the last term is absent when $i = n$. Here Δ denotes the forward difference operators defined by $\Delta x_i = x_{i+1} - x_i$. Putting this last expression into the inequality (5.7), we obtain

$$\mu_2(v,v) \leqslant (\mu_1 + \lambda_2 - \lambda_1)(v,v) - \sum_{i=1}^{n} v_i u_{1,i-1}\Delta\left(\frac{v_{1,i-1}}{u_{1,i-1}}\right)\Delta\left(\frac{u_{2,i-1}}{u_{1,i-1}}\right) - \sum_{i=1}^{n} v_i u_{1,i+1}\Delta\left(\frac{v_{1,i}}{u_{1,i}}\right)\Delta\left(\frac{u_{2,i}}{u_{1,i}}\right),$$

and, using the symmetries of the eigenvectors, we finally obtain

$$\mu_2 - \mu_1 \leqslant \lambda_2 - \lambda_1 - \frac{2}{(v,v)}\sum_{i=1}^{[n/2]} v_i\left[u_{1,i-1}\Delta\left(\frac{v_{1,i-1}}{u_{1,i-1}}\right)\Delta\left(\frac{u_{2,i-1}}{u_{1,i-1}}\right) + u_{1,i+1}\Delta\left(\frac{v_{1,i}}{u_{1,i}}\right)\Delta\left(\frac{u_{2,i}}{u_{1,i}}\right)\right]. \tag{5.8}$$

Here $[n/2]$ denotes the greatest integer less than or equal to $n/2$. The term of index $i = (n+1)/2$ that might appear to be missing from (5.8) for the case of odd n is 0 by the antisymmetry of v.

From (5.8) it is clear that we shall be done if we can show that the summation appearing there is positive. We do this in stages based on Wronskian-type arguments. Before proceeding, we fix our sign convention for v_2 and u_2: we shall always take $v_{2,i}$ and $u_{2,i}$ to be positive for $1 \leqslant i < (n+1)/2$ and negative otherwise. We also need the discrete analogue of a Wronskian, the Casoratian. The Casoratian $C(u,v)$ of two sequences u and v is a new sequence defined by

$$C_i(u,v) = u_i v_{i+1} - u_{i+1} v_i. \tag{5.9}$$

Using the forward difference operator Δ and the recursion relations for u_1 and u_2, we compute

$$\Delta C_i(u_1, u_2) = -(\lambda_2 - \lambda_1) u_{1,i+1} u_{2,i+1}.$$

Summing this from 0 to $j-1$, we obtain an expression for $C_j(u_1, u_2)$:

$$C_j(u_1, u_2) = -(\lambda_2 - \lambda_1) \sum_{i=1}^{j} u_{1,i} u_{2,i} \qquad \text{for} \quad 1 \leqslant j \leqslant n, \tag{5.10}$$

since $C_0(u_1, u_2) = 0$. This relation shows that $C_j(u_1, u_2) < 0$ for $1 \leqslant j \leqslant (n+1)/2$, since $u_{1,j}$ and $u_{2,j}$ are both nonnegative there. [In fact, it is not hard to see that $C_j(u_1, u_2) < 0$ for $1 \leqslant j \leqslant n-1$ by using the sign properties of u_1 and u_2 and the orthogonality condition $\sum_{i=1}^{n} u_{1,i} u_{2,i} = 0$]. Similarly we compute

$$\begin{aligned}
\Delta C_i(u_1, v_1) &= u_{1,i+1}\left[-v_{1,i} + \left(2 + d_{i+1}^0 - \mu_1\right) v_{1,i+1}\right] \\
&\quad - v_{1,i+1}\left[-u_{1,i} + \left(2 + d_{i+1}^1 - \lambda_1\right) u_{1,i+1}\right] - C_i(u_1, v_1) \\
&= -\left(d_{i+1}^1 - d_{i+1}^0 + \mu_1 - \lambda_1\right) u_{1,i+1} v_{1,i+1},
\end{aligned} \tag{5.11}$$

and we arrive at

$$C_j(u_1, v_1) = \sum_{i=1}^{j} \left(d_i^0 - d_i^1 + \lambda_1 - \mu_1\right) u_{1,i} v_{1,i}. \tag{5.12}$$

Now by our assumption on $\mathbf{D}_1 - \mathbf{D}_0$ we know that $d_i^1 - d_i^0$ does not increase with increasing i for $1 \leqslant i \leqslant (n+1)/2$, and hence the expression in parentheses above is nondecreasing on that interval. Since $C_0(u_1, v_1) = 0$ and we can compute

$$\begin{aligned}
C_{n/2}(u_1, v_1) &= u_{1,n/2} v_{1,n/2+1} - v_{1,n/2} u_{1,n/2+1} \\
&= u_{1,n/2} v_{1,n/2} - v_{1,n/2} u_{1,n/2} \\
&= 0
\end{aligned}$$

for n even, and

$$\begin{aligned} C_{(n+1)/2}(u_1, v_1) &= u_{1,(n+1)/2} v_{1,(n+3)/2} - v_{1,(n+1)/2} u_{1,(n+3)/2} \\ &= u_{1,(n+1)/2} v_{1,(n-1)/2} - v_{1,(n+1)/2} u_{1,(n-1)/2} \\ &= -C_{(n-1)/2}(u_1, v_1) \end{aligned}$$

and, by (5.11),

$$\begin{aligned} & C_{(n+1)/2}(u_1, v_1) - C_{(n-1)/2}(u_1, v_1) \\ & \qquad = \left(d^0_{(n+1)/2} - d^1_{(n+1)/2} + \lambda_1 - \mu_1\right) u_{1,(n+1)/2} v_{1,(n+1)/2} \end{aligned}$$

for n odd, we see that either (1) $d_i^1 - d_i^0 = \lambda_1 - \mu_1$ for all $1 \leqslant i \leqslant (n+1)/2$ (and therefore for all $1 \leqslant i \leqslant n$, by symmetry) and hence $C_j(u_1, v_1) = 0$ on this interval or (2) on the interval $0 \leqslant j \leqslant n/2$ $C_j(u_1, v_1)$ starts at 0 and then becomes negative, crossing 0 again precisely at $j = n/2$. In either case we have $C_j(u_1, v_1) \leqslant 0$ for $0 \leqslant j \leqslant n/2$. These facts allow us to finish the proof.

Rewritten in terms of Casoratians, (5.8) becomes

$$\begin{aligned} \mu_2 - \mu_1 \leqslant \lambda_2 - \lambda_1 - \frac{2}{(v,v)} \sum_{i=1}^{[n/2]} \frac{v_i}{u_{1,i}^2} & \left(\frac{C_{i-1}(u_1, v_1) C_{i-1}(u_1, u_2)}{u_{1,i-1}} \right. \\ & \left. + \frac{C_i(u_1, v_1) C_i(u_1, u_2)}{u_{1,i+1}} \right), \end{aligned} \tag{5.13}$$

and since u_1 is positive and $v_i > 0$, $C_i(u_1, u_2) < 0$, and $C_i(u_1, v_1) \leqslant 0$ for $1 \leqslant i \leqslant n/2$, the inequality (5.1) follows and the proof is complete except for the characterization of the case(s) of equality. Since every term in the summation in (5.13) is nonnegative, the only way equality can possibly obtain is for $C_i(u_1, v_1) = 0$ for all i, and this happens if and only if $d_i^1 - d_i^0 \equiv$ constant $= \lambda_1 - \mu_1$ for all i, i.e. $\mathbf{D}_1 - \mathbf{D}_0 \equiv$ constant. It is easy to see directly that equality does obtain in this case, so we are done. ∎

6. AN INEQUALITY BETWEEN THE LOWEST EIGENVALUES OF FOUR MATRICES

In this section we present an inequality between the lowest eigenvalues of four Jacobi matrices $\mathbf{T} = \mathbf{L} + \mathbf{D}$, where now we consider matrices $\mathbf{D}$ that typically have monotonic differences. The ideas and method of proof are very similar to those of Section 5. We shall denote the lowest eigenvalue of $\mathbf{T}$ by $\lambda(\mathbf{D})$ throughout this section. Our main result is as follows.

THEOREM 6.1. *Let* $\mathbf{T}_0 = \mathbf{L} + \mathbf{D}_0$ *and* $\mathbf{T}_i = \mathbf{T}_0 + \mathbf{D}_i$ *for* $i = 1,2,3$ *with* $\mathbf{D}_1$ *nonincreasing*, $\mathbf{D}_2$ *nondecreasing*, *and* $\mathbf{D}_3 = \mathbf{D}_1 + \mathbf{D}_2$. *Then*

$$\lambda(\mathbf{D}_0) + \lambda(\mathbf{D}_0 + \mathbf{D}_3) \geqslant \lambda(\mathbf{D}_0 + \mathbf{D}_1) + \lambda(\mathbf{D}_0 + \mathbf{D}_2). \tag{6.1}$$

Furthermore, equality holds if and only if at least one of $\mathbf{D}_1$ *or* $\mathbf{D}_2$ *is a constant multiple of the* $n \times n$ *identity matrix.*

Proof. The proof follows the same lines as that of Theorem 5.1. One takes as trial vector for u_1 the vector

$$u = \frac{u_0 u_3}{u_2},$$

where u_i denotes the eigenvector for the lowest eigenvalue of the matrix $\mathbf{T}_i$ for each of $i = 0,1,2,3$. The argument proceeds as before to the point where the error term is exhibited as two sums of products of Casoratians. These Casoratians can then be handled exactly as $C_j(u_1, v_1)$ was in the previous section. ■

REMARKS.

(1) In the continuous case the analogue of this result is fundamental to the derivation of gap results for multidimensional Schrödinger operators with spherically symmetric potentials [2, 3]. We do not know of a similar application for the result above.

(2) By using the Rayleigh-Ritz variational principle it is easy to see that (6.1) holds under the weakened hypothesis $\mathbf{D}_3 \geqslant \mathbf{D}_1 + \mathbf{D}_2$ (in the sense of quadratic forms). In this case, if $\mathbf{D}_3 \neq \mathbf{D}_1 + \mathbf{D}_2$, the inequality (6.1) will always be strict.

(3) If $\mathbf{D}_1$ and $\mathbf{D}_2$ are both assumed to be nondecreasing (or nonincreasing), then one obtains the reversed inequality

$$\lambda(\mathbf{D}_0)+\lambda(\mathbf{D}_0+\mathbf{D}_3)\leqslant\lambda(\mathbf{D}_0+\mathbf{D}_1)+\lambda(\mathbf{D}_0+\mathbf{D}_2). \tag{6.2}$$

If we now take $\mathbf{D}_1=\mathbf{D}_2=\frac{1}{2}\mathbf{D}'$, this yields $\lambda(\mathbf{D}_0+\frac{1}{2}\mathbf{D}')\geqslant\frac{1}{2}[\lambda(\mathbf{D}_0)+\lambda(\mathbf{D}_0+\mathbf{D}')]$, which is equivalent to concavity of the ground-state eigenvalue with respect to monotone perturbations. That is, the function $f(t)\equiv\lambda(\mathbf{D}_0+t\mathbf{D}')$ is concave for $t\in\mathfrak{R}$ for any monotone $\mathbf{D}'$ and for any $\mathbf{D}_0$. This is actually just a special case of the well-known result that $\lambda(\mathbf{A}+t\mathbf{B})$ is concave for $t\in\mathfrak{R}$ for any two self-adjoint operators $\mathbf{A}$ and $\mathbf{B}$. However, the results (6.1) and (6.2) are much more general than this, since they apply to any four appropriately related matrices, not just to four matrices lying on a line in the vector space of matrices. The inequality (6.2) is closely related to the notion of a strongly superadditive function as defined in [13]. As the inequality is reversed here, we refer to (6.2) as the property of strong subadditivity for the function λ. In [13], it is shown that convexity does not imply strong superadditivity (equivalently, concavity does not imply strong subadditivity). Thus our inequalities (6.1) and (6.2) in their full generality are distinct from the concavity property of the lowest eigenvalue λ. For further discussion of these inequalities as they relate to Schrödinger operators see [3].

7. FURTHER RESULTS AND CONCLUDING REMARKS

It is not hard to see that results analogous to those for λ_1 in Sections 2 and 3 can be proved for λ_n, the largest eigenvalue of $\mathbf{T}$. The result corresponding to Theorem 3.1 is that

$$\lambda_n(\mathbf{T})\leqslant\lambda_n(\hat{\mathbf{T}}), \tag{7.1}$$

where $\hat{\mathbf{T}}=\mathbf{L}+\hat{\mathbf{D}}$ and $\hat{\mathbf{D}}$ denotes the symmetric-decreasing rearrangement of $\mathbf{D}$. The proof of this rests mainly on a result for the matrix $\mathbf{L}$ analogous to Lemma 2.1, or, perhaps more directly relevant, the result embodied by Equation (2.4). The pertinent problem is to maximize $(x,\mathbf{L}x)$ over $x\in\mathfrak{R}^n$ and all possible rearrangements of x with arbitrary sign changes. The result is that $(x,\mathbf{L}x)$ is maximized by first putting $|x|$ in one of its symmetric-decreasing rearrangements $|x|_R$ or $B(|x|_R)$ and then giving this vector alternating signs. This is easily proved by following the method used in proving Lemma 2.1. Thus we have

$$(x, \mathbf{L}x) \leqslant (s|x|_R, \mathbf{L}(s|x|_R)) \tag{7.2}$$

for all $x \in \mathfrak{R}^n$, where $s = (1, -1, 1, -1, \dots)$ and the multiplication of vectors $s|x|_R$ is to be carried out pointwise. In this formulation, $(x, \mathbf{L}x)$ typically has four maximizers: $s|x|_R, B(s|x|_R)$, and the negatives of these. To carry these results on to the treatment of $\mathbf{T} = \mathbf{L} + \mathbf{D}$, one need only observe that $(sx, \mathbf{D}(sx)) = (x, \mathbf{D}x)$. The inequality (7.1) is then seen to follow easily.

Similarly, one can obtain results about $\lambda_n - \lambda_{n-1}$ corresponding to the results of Section 5 for $\lambda_2 - \lambda_1$ and results about λ_n corresponding to the results of Section 6 for λ_1. The proofs can be carried out in strict analogy to those above and will not be given in detail here. Corresponding to Theorem 5.1 one has that

$$\lambda_n(\mathbf{T}_1) - \lambda_{n-1}(\mathbf{T}_1) \leqslant \lambda_n(\mathbf{T}_0) - \lambda_{n-1}(\mathbf{T}_0) \tag{7.3}$$

when $\mathbf{T}_i = \mathbf{L} + \mathbf{D}_i$, where the $\mathbf{D}_i$ are symmetric diagonal matrices for $i = 0, 1$ and where $\mathbf{D}_1 - \mathbf{D}_0$ is symmetric-increasing. If $\mathbf{D}_1 - \mathbf{D}_0$ is symmetric-decreasing, then the inequality (7.3) is reversed. One also has the analogue of Corollary 5.2 for this case:

$$\lambda_n(\mathbf{T}) - \lambda_{n-1}(\mathbf{T}) \leqslant 2\left[\cos\left(\frac{\pi}{n+1}\right) - \cos\left(\frac{2\pi}{n+1}\right)\right] \tag{7.4}$$

for $\mathbf{T} = \mathbf{L} + \mathbf{D}$ with $\mathbf{D}$ diagonal, symmetric, and symmetric-increasing. If symmetric-increasing is replaced by symmetric-decreasing here, then the inequality (7.4) is reversed.

Corresponding to Theorem 6.1 one has that

$$\lambda_n(\mathbf{D}_0) + \lambda_n(\mathbf{D}_0 + \mathbf{D}_3) \leqslant \lambda_n(\mathbf{D}_0 + \mathbf{D}_1) + \lambda_n(\mathbf{D}_0 + \mathbf{D}_2), \tag{7.5}$$

where $\mathbf{T}_0 = \mathbf{L} + \mathbf{D}_0$ and $\mathbf{T}_i = \mathbf{T}_0 + \mathbf{D}_i$ for $i = 1, 2, 3$ with $\mathbf{D}_1$ and $\mathbf{D}_2$ decreasing and increasing respectively, and with $\mathbf{D}_3 = \mathbf{D}_1 + \mathbf{D}_2$ (or, more generally $\mathbf{D}_3 \leqslant \mathbf{D}_1 + \mathbf{D}_2$). Again, this inequality is reversed if one takes $\mathbf{D}_1$ and $\mathbf{D}_2$ to be either both increasing or both decreasing.

The reversed form of the results (7.3) and (7.4) as well as Theorem 5.1 and Corollary 5.2 could be useful as isolation estimates for the largest or smallest eigenvalues of a matrix $\mathbf{T} = \mathbf{L} + \mathbf{D}$ in preparation for using an iterative scheme for determining the eigenvalue λ_n or λ_1 and its corresponding eigenvector. The gap estimate would give one an *a priori* estimate for the efficiency of the method.

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